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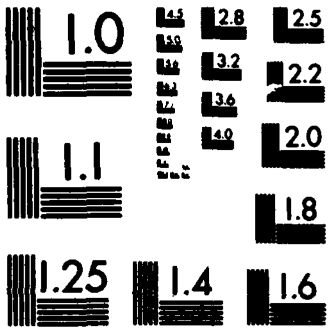

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During this period the principal investigator has considered the effect of the artificial far field boundary on a finite different approximation of the Navier-Stokes equations for a compressible fluid considered for two dimensional flows. It was shown in this research effort that the incorrect choice of boundary conditions for the far-field boundary can result in some of the following phenomena: (1) reflecting boundary conditions, in which the disturbances in the physical variables represented by the difference between the steady state flow and the initial conditions are not allowed to (CONTINUED)

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ITEM #20, CONT.: exit through the far-field boundary but instead continue echoing up and down the grid; (2) under-specified boundary conditions, in which the converged steady state solution may depend on the initial conditions; and (3) over-specified boundary conditions in which large errors are introduced before convergence takes place. The following conclusions may be drawn from the numerical experiments conducted as a part of the research effort: There are four distinct artificial nonreflecting boundaries containing free-stream information, these being the inflow, outflow, top-sidewall and bottom-sidewall. Each of these boundaries must be treated differentially, or errors are introduced into the system. The final report received for this effort contains an extensive section which it is expected will be submitted as a scientific paper to a major refereed journal.

AFOSR-TR- 83-0928

FINAL REPORT

Grant #

AFOSR 82 0171

P. J. McKenna
Principal Investigator

83 11 29 211

The flow of fluid around obstacles in two dimensions is described by the compressible Navier Stokes equations

$$\frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{1}$$

where

$$u = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix} \qquad E = \begin{bmatrix} \rho u \\ \rho u^2 - \sigma_{xx} \\ \rho uv - \zeta_{xy} \\ \rho ue - u\sigma_{xx} - v\zeta_{xy} - q_x \end{bmatrix}$$

$$F = \begin{bmatrix} \rho v \\ \rho uv - \zeta_{xy} \\ \rho v^2 - \sigma_{yy} \\ \rho ve - v\sigma_{yy} - u\zeta_{xy} - q_y \end{bmatrix}$$

$$\sigma_{xx} = - p - (2/3)\ m\vec{\nabla}.\vec{u} + 2mu_x \qquad q_x = kT_x$$

$$\zeta_{xy} = m(u_y + v_x)$$

$$\sigma_{yy} = - p - (2/3)\ m\vec{\nabla}.\vec{u} + 2mu_x \qquad q_y = kT_y$$

$$p = PRT \quad m = m(T) \qquad e = cvT + \frac{u^2 + v^2}{2}$$

where the four variables ρ, u, v e represent the physical quantities of density, x- and y- components of velocity, and internal energy.

This nonlinear system of mixed parabolic-hyperbolic type in two space dimensions and time with four independent variables must be solved in an exterior region in R^2. The geometry will depend on the particular physical situation that one is attempting to model.

The situation we shall be interested in occurs in modelling flight conditions, in which the conditions at infinity are prescribed with a large u-velocity and v-velocity zero and prescibed ρ_∞ and e_∞. Fluid flows around an obstacle in x-y space.

Usually, this equation is solved numerically using a finite difference scheme of the Lax-Weudroff type, such as the MacCormack ADE mettrod. Since these calculations can only be made on a finite grid of points in xy space, an artificial far-field boundary is created. This boundary ought be sufficiently far away from the object around which the fluid is flowing so that local phenomena are not omitted by omitting part of the region of fluid flow. On the other hand, the farther away the region is, the more grid-points need to be included and thus the more expensive time-consuming the computations become.

One then has the problem of deciding what effect this new boundary has on the solution of the problem. Because of the viscosity terms in (1) and the additional artificial viscosity introduced by the finite difference schemes, some boundary conditions must be imposed.

As we shall show in this report on numerical experiments, cosiderable care must be exercised in the choice of the boundary

conditions. If one is interested in steady state flow, then one starts off with a bad approximation, and hopes that the errors in the numerical solution propagate out of the region as transitory disturbances in the physical variables. One then expects to converge to the steady state flow.

We shall show in this paper that the incorrect choice of boundary conditions can give rise to some of the following phenomena: i) reflecting boundary conditions, in which the disturbances in the physical variables represented by the difference between the steady state flow and the initial conditions are not allowed to exit through the far-field boundary but instead continue echoing up and down the grid, and giving rise to spurious oscillatory solutions; ii) under-specified boundary conditions, in which the converged steady state solution may depend on the initial conditions; iii) overspecified boundary conditions in which large errors are introduced before convergence takes place.

In this paper, we first discuss the theory for simple linear hyperbolic systems in one space dimension. We then analyse several computational experiments in the light of this theory in one space dimension. This show up the phenomena which we wish to discuss in their simplest setting. Finally we discuss the more complicated setting of two space dimensions and point out how the phenomena of one space dimension can be recognized in this more complicated setting.

A REVIEW OF THE LINEAR CASE

The method used in the calculations which are the subject of this paper is the MacCormack alternating direction explicit scheme. [6] [15]. This is a multistep efficient scheme which reduces in the linear case to the Lax-Wendroff scheme. For a diagonal N x N matrix A, this scheme approximates the equation $U_t + AU_x = 0$ by.

(2) display
$$\left[\begin{array}{l} u_j^{n+1} = u_j^n - A\frac{\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n). \\ \qquad + A^2\left(\frac{\Delta t}{\Delta x}\right)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n) \end{array}\right.$$

As usual, $\{j_{1<j<J}\}$ represents the space step and n represents the time steps. If we are considering the equation

$$U_t + AU_x = 0$$

on the region $\{(x,t),\ 0<x<1,\ t\geq 0\}$ then the analytic solution is determined by the initial conditions and boundary conditions at $x = 0$ and $x = 1$. If the first k eigenvalues are positive and the remaining N - k are negative than the quantities u_1---u_k must be prescribed at $x = 0$ and u_{k+1}---u_N must be prescribed at $x = 1$. Thus if $W_I = (u_1, u_2, \text{---} u_k, 0\ 0\text{---}0)$ and $W_{II} =$ $(0, 0, \text{---}0, u_{k+1}\text{---}u_N)$ then for well-posedness, the boundary conditions must be

$$W_I = f(t) + B_0 W_{II} \qquad \text{at } x = 0$$
$$W_{II} = g(t) + B_1 W_I \qquad \text{at } x = 1.$$

This gives a total of N boundary conditions. If either kx(n-k) matrix B_1 and the (n-k)xk matrix B_0 are non-zero, then the boundaries are reflecting, that is a wave in W_I travelling left to right will be reflected as a wave in W_{II} running right to

left.

If a boundary is supposed to be non-physical, then it should not be reflecting, since the reflections would depend on the location of the artificial or numerical boundary.

Now let us consider the difference scheme of (2). If the grid points are given by $\{x_j\}_0^J$ with $x = 0$ and $x_J = 1$, then it is clear from (2) that 2N boundary conditions are required. Thus we must prescribe boundary conditions in such a way as to least affect the closeness of the numerical to the analytic solution.

This situation has been the subject of several papers. In [3], Gustaffson and Kreiss point out the danger of over-specification. By this is meant that all u_i are specified at both endpoints. This might be tempting because one might argue that if the boundary conditions u_i---u_k are given constants at x = 0, then eventually these same values will be assumed by these variables at x = 1. However, in [4] it is pointed out that convergence may or may not occur, depending on whether the number of grid points is odd or even.

A method which works well as pointed out in [3] is to impose

$$\frac{\partial u_i}{\partial x}(1,t) = 0 \qquad 1 \leq i \leq k.$$

$$\frac{\partial u_i}{\partial x}(o,t) = 0 \qquad k+1 \leq i \leq n.$$

or in the numerical scheme

$$u_{i,J} = u_{i,J-1} \qquad 1 \leq i \leq k.$$

$$u_{i,o} = u_{i,1} \qquad k+1 \leq i \leq n.$$

This introduces small errors at the outflow but these errors do not propogate upstream. This is proved analytically by Parter [10].

In [12], many different numerical boundary conditions are given. The conclusion is that upwind differencing at the point of outflow

$$u_J^{n+1} = u_j^n + \lambda i\left(\frac{\Delta t}{\Delta x}\right)(u_{J-1}-u_J)$$

is most accurate, although it converges with the same speed as the previously discussed $u_x = 0$.

The most serious error which one could make would be to prescribe conditions at the wrong end. In other words, since u_1 is right running, this would involve prescribing u_1 at $x = 1$ and imposing $u_{1x} = 1$ at $x = 0$. This would result in convergence to a steady state which depends on the initial conditions.

THE NAVIER-STOKES EQUATIONS AND CHARACTERISTIC VARIABLES:

We now begin our discussion of the equations of gas dynamics. We will neglect viscosity for the puposes of this analysis. We will assume that the flow is one-dimensional and subsonic and that the deviations from free-stream solutions are small. This will allow us to neglect second order terms.

There are many forms of this equation, but the one most suitable for the present discussion is

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)\ u & \gamma - 1 \\ (\gamma-1)4^3 - \frac{\gamma e u}{\rho} & \frac{\gamma e}{\rho} - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{pmatrix}$$

and

$$U = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}$$

or in terms of physical variable

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0$$

where

$$A = M^{-1}\ AM$$

and

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -u/p & 1/p & U \\ (\frac{\gamma-1}{2})u^2 & (1-\gamma)u & (\gamma-1) \end{pmatrix}$$

Here we make the key assumption that deviations from the free stream are going to be sufficiently small that we can treat the entries in the matrix A as being approximately constant (at least locally). Denote these frozen variables by 0-subscript. We then make the substitution

(4)
$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/c_o^2 \\ 0 & 1 & 1/\rho_o c_o \\ 0 & -1 & 1/\rho_o c_o \end{pmatrix} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}$$

and when this is substituted into (6) we obtain

$$(5)\qquad \frac{\partial W_1}{\partial t} + u_o \frac{\partial W_1}{\partial x} = 0 \qquad W_1 = \rho - \frac{1}{c^2}p$$

$$\frac{\partial W_2}{\partial t} + (u+c_o)\frac{\partial W_2}{\partial W} = 0 \qquad W_2 = u + \frac{1}{\rho_o c_o}p$$

$$\frac{\partial W_3}{\partial t} + (u_o-c_o)\frac{\partial W_3}{\partial x} = 0 \qquad W_3 = -u + \frac{1}{\rho_o c_o}p$$

Notice now how this breaks down into two separate cases. On the one hand, if flow is supersonic then all wave motion is in the left to right direction. In this case all analytic bound y condition sought be prescribed at the left hand side and ' y numerical boundary conditions prescribed at the right han e.

Since the substitution (4) is equivalent to

$$\rho = K_1 + (\rho_o/2c_o)(K_2+K_3)$$
$$u = 1/2(K_2-K_3)$$
$$p = \rho_o c_o/2(K_2+K_3)$$

it follows that prescribing all physical variables at the inflow and prescribing $\partial\rho/\partial x = \partial u/\partial x = \partial p/\partial x = 0$ at the outflow is legitimate in terms of analytical and numerical requirements in the supersonic case.

However, we must now consider the case of subsonic flow. In this case the situation is completely different. Here, two of the variables W_1 and W_2 go left to right with velocities u and u+c respectively, whereas one of the variables runs right to left with velocity c_o-u_o. While the variables W_2 and W_3 have no clear physical significance, yet it is only be considering these variables that the full wave structure of the equations (5) or (6) can be understood. Thus, one would be led to predict, for small

deviations from free stream conditions, that the best boundary conditions would be for an interval (o,L)

$$W_1\ (o,t) = K_1 \qquad \frac{dW_1}{dx}\ (L,t) = 0$$

$$W_2\ (o,t) = K_2 \qquad \frac{dW_2}{dx}\ (L,t) = 0 \tag{6}$$

$$\frac{dW_3}{dx}\ (o,t) = 0 \qquad W_3\ (L,t) = K_3$$

Note the curious aspect of these boundary conditions. In order to prescribe the numerical values K_1 and K_2, we need to know accurately all three physical variables at some distance to the left. However, only the two combinations K_1 and K_2 are prescribed. This can be summarized by saying that while we have used all three pieces of information upstream, we have done so in such a way that one degree of freedom remains, thus allowing the waves in W_3 to exit without reflections.

On the basis of the linearized model, various other combinations would be well-posed. For example, it is possible to prescribe K_3 in terms of either K_1 or K_2 at the outflow x = L. Thus at the outflow one may prescribe

$$W_2\ (L,t) = F_3\ (t) + c_1 W_1\ (L,t) + c_2 W_2\ (L,t)$$

For example if $c_1 = 0$, $c_2 = 1$, then this amounts to putting

$$u\ (L,t) = 1/2\ F_3 \tag{7}$$

i.e., we prescribe velocity at the outflow.

Alternatively, we might take $c_1 = 0$, $c_2 = -1$ and we would get

$$p\ (L,t) = ((p_o c_o)/2) F_3 \tag{8}$$

i.e. we prescribe pressure at the outflow. Many other combi

nations are possible, but as remarked in section II, all these will cause errors in the initial data to be reflected back into the medium as waves running from right to left. For example, we would predict that an error in W_3 would be reflected back as an error in W_1 if we use boundary condition (8). As we shall see, this is exactly what happens.

At the inflow end, we may prescribe W_1 and W_2 in terms of W_3. Thus the following boundary conditions are well posed;

$$W_1\ (o,t) = F_1 + c_1 W_3\ (o,t) \qquad (9)$$

$$W_2\ (o,t) = F_2 + c_2 W_3\ (o,t)$$

For example, choosing $c_2 = +1$ in (11b) corresponds to

$$u(o,t) = (1/2)F_2$$

(i.e. prescribing u at the inflow) and $c_2 = -1$ corresponds to

$$p(o,t) = (\rho_o c_o/2)F_2$$

(i.e. prescribing p). One can prescribe the combination (u,ρ) by first choosing $c_2 = 1$ (thereby prescribing u) and then choosing $c_1 = \rho_o/c_o$, thereby prescribing p in terms of a given F, and a prescribed u(o,t). Since (11a) and (11b) reduce to

$$u(1+c_2) + \frac{p}{\rho_o c_o}(1-c_2) = F_2$$

$$\rho - \frac{p}{\rho_o c_o}(\rho_o/c_o - c_1) + c_1 u = F_1$$

about the only condition we cannot prescribe is u(o,t), p(o,t), since there is no choice of c_1, c_2 to eliminate ρ from these equations.

Again, we emphasize that each of these boundary conditions is <u>reflecting</u>, i.e. deviations from the free stream in the initial data get reflected back as waves in W_1 and W_2 and then

travel back downstream. About the worst thing that can be done is to prescribe reflecting boundary conditions at the inflow $x = 0$ and the outflow $x = L$. In this case errors can keep being reflected up and down the region, never being allowed to exit. This prevents convergence to a steady state and may even give rise to fictitious periodic oscillations.

We conclude this section with a review of the conclusions on boundary conditions. Based on the linear model, boundary conditions (8) seem optimal. Any other prescription of the physical variables, although well posed, casues reflections of the deviation from the true solution. If for example, only the physical variable p_∞ is known at the outflow then it is possible to prescribe p at the outflow, in such a manner that the problem remains well posed. We emphasize that this will cause errors to propagate upstream, thereby slowing the process of convergence to steady state. This may not be too bad, so long as the upstream boundary conditions are not also reflecting. On the other hand, if they are, then convergence to free stream may never occur.

VI. DISCUSSION OF NUMERICAL RESULTS - NONLINEAR COUPLING:

The one dimensional Navier-Stokes equations were solved with an alternating direction explicit MacCormack scheme, on a one-dimensional net with forty grid points. The code was an exact one-dimensional version of a three-dimensionsal code which had proved successful in many supersonic studies [6], [11]. There were two questions to answer. The first was, given that equation (7) is correct for infinitesimally small deviations from a

constant free steam, how correct is it when deviations of an intermediate (of the order of 10%) magnitude are present instead? It would be too much to hope that the variables W_1, W_2, and W_3, remain uncoupled, but we should be able to get an idea of the order of magnitudes of the coupling involved. The second problem of course, is to assess the influence of the various types of boundary conditions commonly employed. We will deal with the latter problem in section VII.

To do this, we considered a uniform free stream situation, with pressure equal to 2000 lbs/ft^2, velocity equal to 548 ft/sec and density equal to 0.0023 slugs/ft^3. We created a deviation from the steady state condition in a variety of ways as in [9], [10], and then watched the progress (or lack of it) to a steady state. A variety different wrong initial conditions were used. One type was to impose a 10% deviation in one of the variables K_1, K_2, K_3 at the points $\{x_j, j=18,19,20,21,22\}$. We should then watch the disturbance, graphically as it propagated up or down stream. Another possiblity was to put in uniformly wrong initial conditions where some of all of the characteristic variables W_1, W_2, W_3 are perturbed throughout by a percentage error of 10%. Each plot then showed the percentage error, with the different curves representing the progress of time as one ascends the plot. The curves are plotted every twenty five time steps when $\Delta t = (.9)(\Delta x)/(u+c)$. We also point out the percentage errors in W_1, W_2, W_3 at the end of the run, so as to obtain

information on relative accuracy and speed of convergence of the various methods.

Figure 1 gives the results of an experiment, which is ideal in terms of the linear theory. An initial disturbance in W_3 the left-running characteristic variable is given and we observe deflections in the variables W_1, W_2, W_3. As one can readily see from the pictures, the disturbance propagates upstream rapidly until convergence is reached (.1% agreement with physical variables), which takes place within 130 time steps. This gives us six curves. Note that although a good deal of undershoot and overshoot in W_3 becomes apparent, there is no significant interaction with W_1 or W_2. The same situation appears with initial disturbances in W_1, the slow-moving right running wave. From this experiment it appears that deviations in either W_1 or W_3 will not effect either of the other two variables. However, as shown in Figures 2 and 3, when initial disturbances are in W_2, a different situation exists. Figure 2 shows a uniform initial disturbance in W_2 of minus ten percent, while W_1 and W_3 are left undisturbed. Initially the wave in W_2, propagates rapidly out of the medium. Indeed, after fifty time steps it is essentially gone from the picture. However, this does no happen without affecting the of the two variables. Notice how, in the top graph, a large disturbance is left in W_1 after fifty time steps and in W_3, we have that W_3 values one almost constant at minus eight percent. However, once the W_2 wave has made its exit, the other two variables uncouple and resume their normal wave motion, and the error can be seen propagating out of the solution in the

usual way convergence is attained within three hundred interations. These particular results illustrate what became increasingly clear throughout the series; perturbations in W_2 had a large effect on W_1 and W_3 whereas if W_2 was not perturbed, W_1 and W_3 behaved as if uncoupled. Perturbations in W_1 and W_3 had little effect on W_2. No qualitative explanation of the phenomenon is know at this time. Figure 3 shows the same phenomenon. Here an error of -10% is made in W_2 whereas errors of +10% are made in W_1 and W_3. Again, we see that until the W_2 wave exits, there are massive disturbances in the wave structive of W_1 and W_2. As soon as W_2 exits, (after 75 interations) the regular wave structive reasserts itself and errors propagate out in predictable wave-

like manner. Again, convergence takes approximately three hundred and twenty five iterations. It seems clear that this is optimal given the limited wave velocity, so we can deduce that these affects are due to the nonlinear coupling. Thus, even after the W_2 wave exists it will take at least a maximum time of $\{L/(u-c), L/u\}$ seconds for the resulting errors to propagate out of the system.

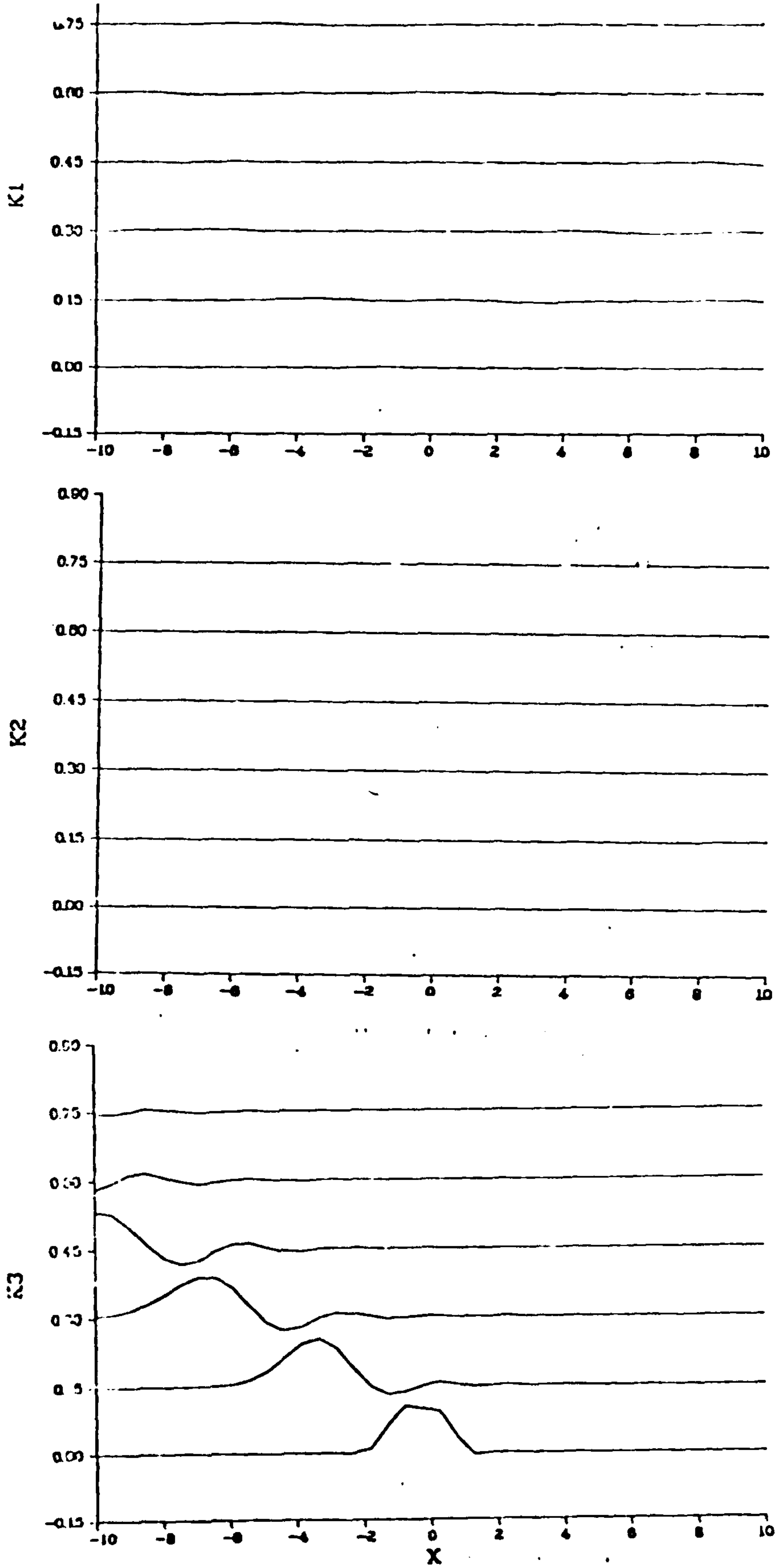


FIGURE 1
A left-running characteristic variable

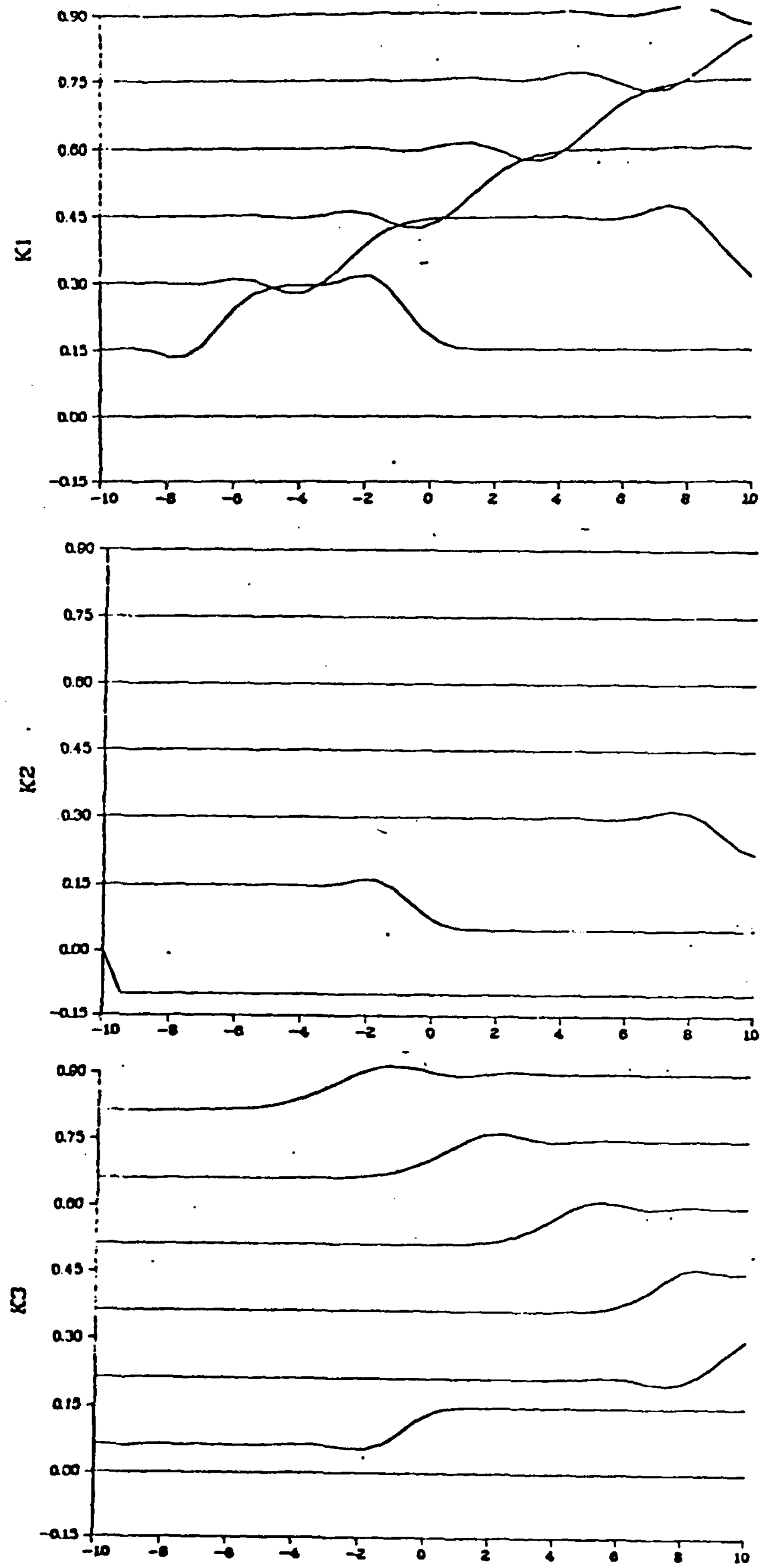


FIGURE 2
Nonlinear coupling: until the disturbance in the second variable exits, it disturbs the other two.

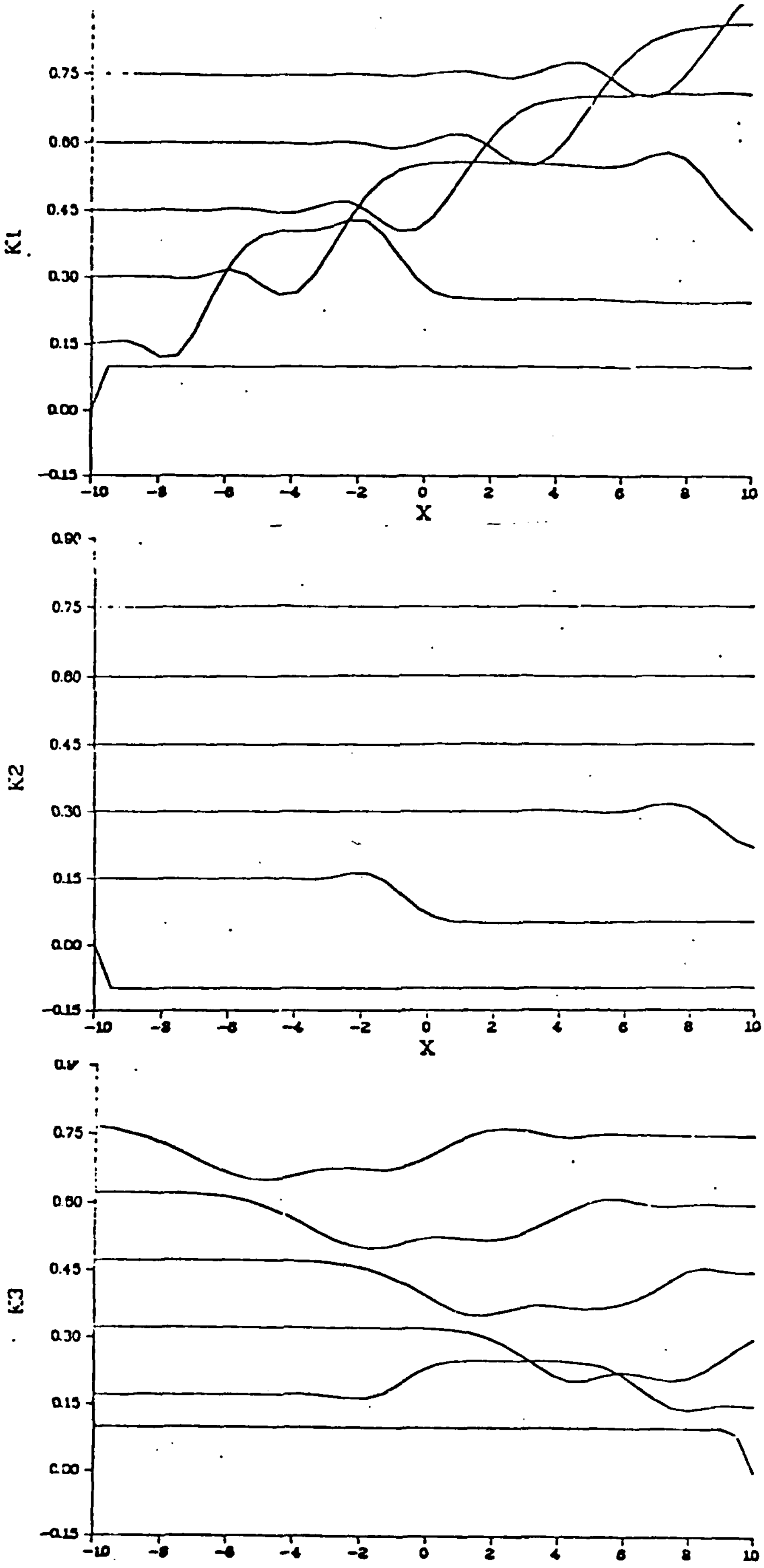


FIGURE 3
More evidence of nonlinear coupling, then the first and third variables behave as if uncoupled.

VII. DISCUSSION OF NUMERICAL RESULTS - BOUNDARY CONDITIONS:

The linear "small deflection" theory predicts that the best boundary conditions would be the prescription of the characteristic variables at their point of entry with some form of (stable) numerical boundary condition for the point of exit. Here are two such schemes

INFLOW	OUTFLOW	
$W_1 \ (o,t) = K_1$	$\frac{dW_1}{dx} \ (L,t) = 0$	
$W_2 \ (o,t) = K_2$	$\frac{dW_2}{dx} \ (L,t) = 0$	(12)
$\frac{dW_3}{dx} \ (o,t) = 0$	$W_3 \ (L,t) = K_3$	

Here K_1 and K_2 are numbers calculated from the known values of u, p, ρ at the inflow and K_3 is calculated from the known values of u and p at the outflow. Notice, however, the one "degree of freedom" is left at the inflow point. This allows the variables to adjust but in compensating ways. The boundary conditions in the code are usually in terms of the physical variables so we translate (12) to physical variables.

INFLOW

$$P_1 = \frac{\rho_o c_o}{2} \ [K_2 \ u_2 - (1/\rho_o c_o) p_2]$$

$$u_1 = 1/2 \ [K_2 + u_2 - (1/\rho_o c_o) p_2]$$

$$\rho_1 = K_1 + (\rho_o/2c_o)[K_2 - u_2 + (1/\rho_o c_o) p_2]$$

OUTFLOW

$$u_N = 1/2[u_{n-1} + (1/\rho_o c_o) \ p_{N-1} + K_3$$

$$P_N = (\rho_o c_o/2)[K_3 + u_{N-1} + (1/\rho_o c_o) \ P_{N-1}]$$

$$\rho_N = (\rho_o/2c_o)[K_3 + u_{N-1}] - [1/(2c_o^2)] \ P_{N-1} + \rho_{N-1}$$

This set of boundary conditions is predicted to work well in the linear studies of one equation, occurring in [2] and [3]. We shall call these boundary conditions the "no-change characteristic boundary conditions". Another possiblity, suggested by one-D analogues in [2], is the following:

INFLOW / OUTFLOW

$$W_1\,(o,t) = K_1 \qquad\qquad \frac{\partial W_1}{\partial t} + u_N \frac{\partial W_1}{\partial x} = 0$$

$$W_2\,(o,t) = K_2 \qquad\qquad \frac{\partial W_2}{\partial t} + (u-c)_N \frac{\partial W_2}{\partial x} = 0 \qquad (13)$$

$$\frac{\partial W_3}{\partial t} + (u-c)_1 \frac{\partial W_3}{\partial x} = 0 \qquad\qquad W_3\,(L,t) = K_3$$

where derivaties in the x-variable are downwind at the inflow and upwind at the outflow and forward in time. The numbers K_1, K_2, K_3 are prescribed as before. In terms of the physical variables these translate into

INFLOW

$$u_1^{n+1} = 1/2[K_2 + u_1^n - (1/\rho_o c_o)p_1^n + (u_o-c_o)(\tfrac{\Delta t}{\Delta x})[u_1^n - u_2^n + (1/\rho_o c_o)(p_2^n + p_1^n)]]$$

$$p_1^{n+1} = (\rho_o c_o/2)[K_2 + (1/\rho_o c_o)p_1^n - u_1^n(u_o-c_o)(\Delta t/\Delta x)[u_2^n u_1^n + (1/\rho_o c_o)(p_1^n + p_2^n)]]$$

$$\rho_1^{n+1} = K_1 + (\rho_o/2c_o)[K_2 + (1/\rho_o c_o)p_1^n - u_1^n + (u_o-c_o)(\Delta t/\Delta x)[u_2^n - u_1^n + (1/\rho_o c_o)(p_1^n - p_2^n)]$$

OUTFLOW

$$u_1^{+1} = (1/2)[u_N^n + \frac{1}{\rho_o c_o}\,p_N^n - K_3 + (\Delta t/\Delta x)(u_o+c_o)[u_{N-1}^n - u_N^n + (1/\rho_o c_o)(p_{N-1}^n + p_N^n)]]$$

$$p_N^n = (\rho_o c_o/2)[K_3 + u_N^n + (1/\rho_o c_o)p_N^n + (\Delta t/\Delta x)(u_o+c_o)[u_{N-1}^n - u_N^n + (1/\rho_o c_o)(p_{N-1}^n - p_N^n]$$

$$\rho_N^{n+1} = (\rho_o/2c_o)[K_3 + u_N^n + (1/\rho_o c_o)p_N^n + (\Delta t/\Delta x)(u_o+c_o)u_{N-1}^n - u_N^n + 1/\rho_o c_o p_{N-1}^n - p_N^n]]$$
$$+ \rho_N^n - (1/c_o^2)p_N^n + (\Delta t/\Delta x)u_o[p_{N-1}^n - \rho_N^n + (1/c_o^2)(p_N^n - p_{N-1}^n)]$$

We shall call these the "windward difference characteristic boundary conditions". Note: u_o, c_o can be different values at

inflow and outflow. The performance of the code with either of these was analyzed by posing initial conditions in which there was a disturbance in one or more of the characteristic variables either locally at the center of the grid or uniformly throughout the grid, of the order of 10%.

Thus figure 1 shows what happens if the disturbance in only in the third characteristics variable locally using the windward characteristic variables.

Figure 3 shows the effect of a plus +10% error in the initial conditions W_1 and W_3 and a -10% error in W_2. (The first curve from the bottom is the initial state of th variable, and the others are the states at intervals of 25 interations). In figure 2, we have an initial disturbance in W_2 of -10% with no initial disturbance in W_1 or W_3. The pictures look essentiallly the same as figure 1. There is considerable nonlinear interaction until the W_2 wave exists, and then uncoupled wave motion to the right in the first variable (W_1) and to the left in the third variable (W_3). There are no reflections when the W_1 and W_3 waves exit and convergence is reached in 300 iterations.

These computations were made using the windward differencing characteristic boundary conditions, although the same results were obtained with the no change characteristic boundary conditions.

In Figure 4, we show the effect of a local disturbance at the center of the grid in the W_2 variable with the second set of B.C.'s and in figure 5 we show a speeded up version (every 50 iterations) of the same disturbance with the first set of B.C.'s.

The third and fifth graphs on figure 4 are almost exactly the same as the second and fourth on figure 5. Figure 5 shows convergence being reached in 300 iterations.

We conclude that either of the first two sets of boundary conditions give optimal convergence since convergence cannot take place until the wave in W_2 exists (very quickly) and the residual (nonlinear) effects of W_2 on W_3 can exit upstream. If they can do this without any reflections, then the convergence is essentially optimal.

Sometimes, it is objected that in a wind tunnel experiment, the only variable known downstream is pressure and that we are requiring too much information in prescribing K_3, which demands a knowledge of p and u at the outflow. Suppose, then, we just prescribe p_∞ at the outflow using the otherwise successful conditions of $\frac{\partial W_1}{\partial x} = \frac{\partial W_2}{\partial x} = 0$ as complementary numerical boundary conditions. Then we could have, for example

INFLOW	OUTFLOW	
$W_1\ (o,t) = K_1$	$p = p_\infty$	
$W_2(o,t) = K_2$	$\frac{dW_1}{dx}\ (L,t) = 0$	(14)
$\frac{dW_3}{dx} = (o,t) = 0$	$\frac{dW_2}{dx} = (L,t) = 0$	

The inflow boundary conditions are precisely those of (12). The outflow boundary conditions (used by Steger [12]) are

$$p_N = p_\infty$$
$$\rho_N = (1/c_o^2)(p_\infty - p_{N-1}) + \rho_{N-1}$$
$$u_N = (1/\rho_o c_o)(p_{N-1} - p_\infty) + u_{N-1}$$

The predictions of section VI are clear. The fact that p is prescribed means that when a wave in W_2 comes downstream, it exits by adjusting the u values at $x = x_N$. This in turn causes disturbances in $W_3 = -u + (1/\rho_o c_o)p_\infty$ which cause a reflected wave upstream. This wave can be seen by comparing figure (6) with figure (4). Since the upstream B.C. is non-reflecting, this means that the left running wave will exit without incidence. When composed with boundary conditions (12) or (13) it is obviously less desirable because of the magnitude of the reflection in W_3. However it does converge in approximately 300 iterations, which is again almost optimal. The effect of these large oscillations in more complicated geometries may prove undesirable, however.

We briefly review our progress so far. Two sets of non-reflecting boundary conditions have been produced, both of which give optimal convergence but which rely on a great deal of information at both ends. The information however is used in such a way as to allow additional degrees of freedom for the waves to exit without repeated reflections. One reflecting and non-reflecting boundary condition can be combined to obtain almost optimal convergence at the cost of some large left running reflections, whose effect in more complicated geometries remains uncertain. While B.C. (12) was expected to be less accurate than B.C. (13), little evidence for this has been uncovered, except at

the boundaries. Time dependent periodic flows (e.g. self excited oscillations) may prefer B.C. (13) however. We now consider some of the other boundary conditions which have been tried previously in the literature.

First we consider the case of reflecting boundary conditions. These occur in several places in the literature, for example in [10] and [12] and have been discussed in section VI. As we have seen, these arise from prescibing combinations of characteristic variables such as pressure downstream, and other combinations (perhaps to density and velocity upstream). In [10] Rudy and Strikwerda considered (among many others) the boundary conditions

INFLOW	OUTFLOW	
$u = u_\infty$	$\frac{du}{dx} = 0$	
$T = T_\infty$	$\frac{d\rho}{dx} = 0$	(15)
$\frac{dW_3}{dx} = 0$	$p = p_\infty$	

and in [12] Steger uses

$u = u_\infty$	$\frac{dW_1}{dx} = 0$	
$\rho = \rho_\infty$	$\frac{dW_1}{dx} = 0$	(16)
$\frac{dW_3}{dx} = 0$	$p = p_\infty$	

Figure 7 show the effects of an initial local disturbance in W_3 on both of these sets of boundary conditions, (13) on the left and (14) on the right. Note that first there is only a disturbance in the bottom picture. By iteration 75 (fourth curve up from the bottom) we can see reflections in both W_2 and W_1 although the W_2 deiration is a little harder to see. By

iteration 125 it can be seen that the W_2 wave has travelled downstream and is reflected back upstream in W_3. These reflections continued (somewhere smeared out) for a least 2000 iterations. Perhaps most startling is (at least at the beginning) how similar they are. A conclusion may be drawn from B.C.'s (14), (15), and (16). Prescribing physical values instead of characteristic values gives rise to reflections. If the reflections can occur only at one end, this does not impede convergence. However, reflecting conditions at both ends can be disastrous. This accounts for the "spurious pressure waves" mentioned by Moretti [7].

The case of prescription at the wrong end is now considered. In this case, the variables u, p, ρ, are prescribed at the inflow and the conditious

$$\frac{du}{dx} = \frac{d\rho}{dx} = \frac{dp}{dx} = 0$$

are prescribed at the outflow. In this case, square wave disturbances which effected only the interiors of the domain exited as in figure 1, with some minor oscillations. However, when a uniformly wrong initial condition was imposed, convergence was very slow with large oscillations, and the solution converged to the wrong values, with errors of as much as 27%. The converged value was a function of the initial condition at the outflow end, as predicted in Gustafson and Kreiss [3]. The resulting graphs are given in figure 8.

Related to the above problem is the method of over prescription of boundaries. This method is mentioned in [9] as giving good results although the authors caution against it on

the grounds of small oscillations being present. In fact the situation is much more serious. If initial waves in the interior of the domain are used with the initial conditions correct near the outflow, then the solution converges rapidly as the travelling waves exit without reflections and with minor oscillations. However, if the initial data is uniformaly wrong with an error initially at the outflow point, the W_1 and W_2 converge rapidly but W_3 accumulates huge errors of the order of 70%. Eventually when W_1 and W_2 are converged, the correct value is propagated upwind in W_3 but taking large amounts of time to converge because of the large errors near the inflow point. Indeed, as W_2 becomes more accurate downstream the inaccuracies become much larger (140%) upstream. Particularly in a time dependent problem or in a problem with more complicated geometries, this could be truly disastrous. It points to another fact: if a new boundary condition is being tested, it is not sufficient to consider initial value perturbations from free stream which are non-zero in the interior only. In this case, we might have drawn totally wrong conclusions from the time taken for convergence.

In [10], a separate non-reflecting boundary conditrion is proposed. This boundary conditon alters the value of K_3, increasing it if the computed value of p is less than p_∞, and decreasing it if the computed value of p is greater than p_∞. Some thought shows that this cannot be optimal. Indeed, we can choose a variation from the initial conditional in which p is less than p_∞, but because u is smaller than u_∞, the computed W_3 is actually larger than $W_{3,\infty}$. In this case, the proposed non-

reflecting boundary condition of [10] could introduce more errors into the system, by increasing W_3 at the outflow point.

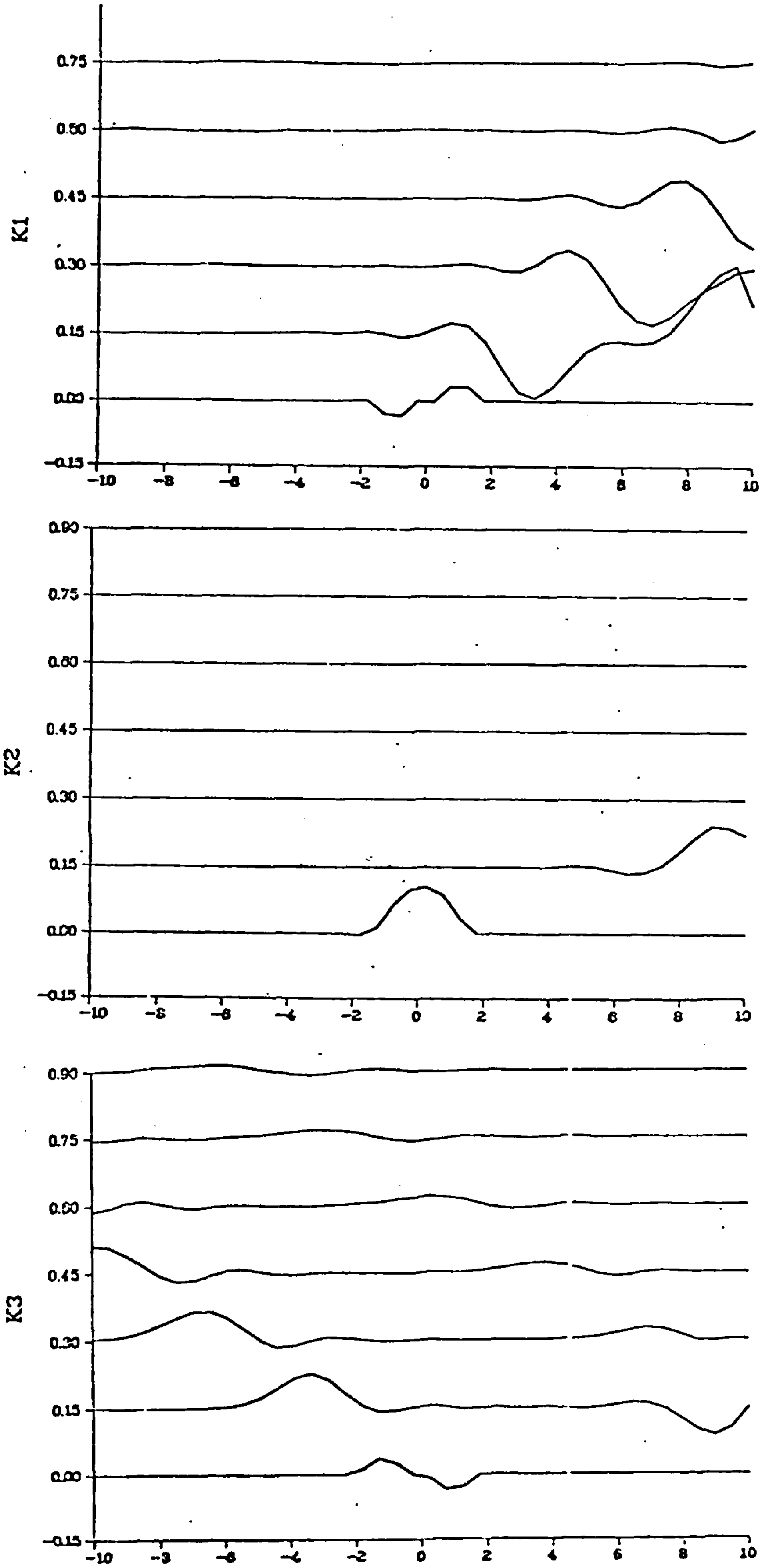


FIGURE 4
A square wave in W_2 exits, leaving trailing disturbances but no reflection. See figure (6) if outflow B.C. is reflecting.

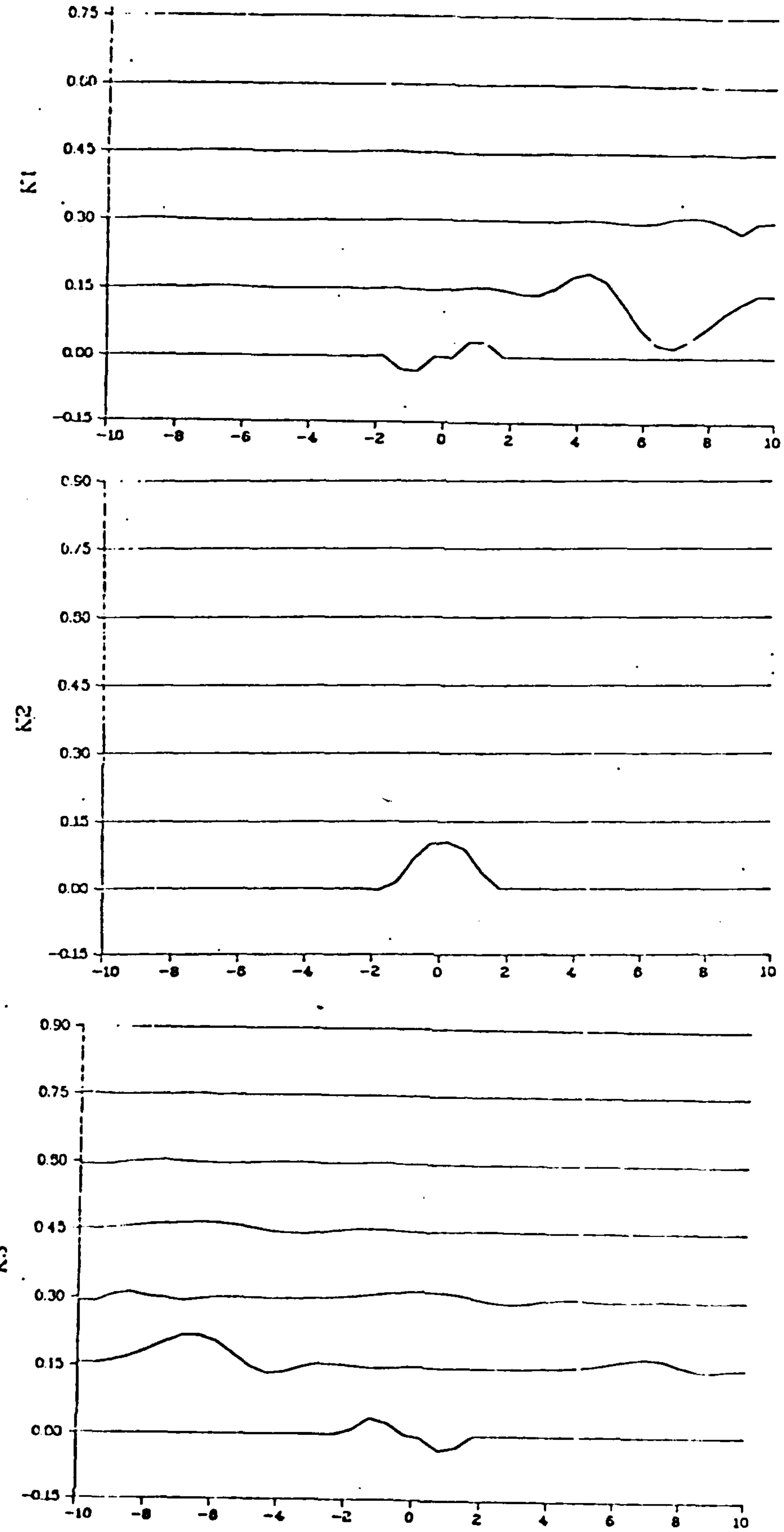


FIGURE 5
A speeded up version of figure 4.

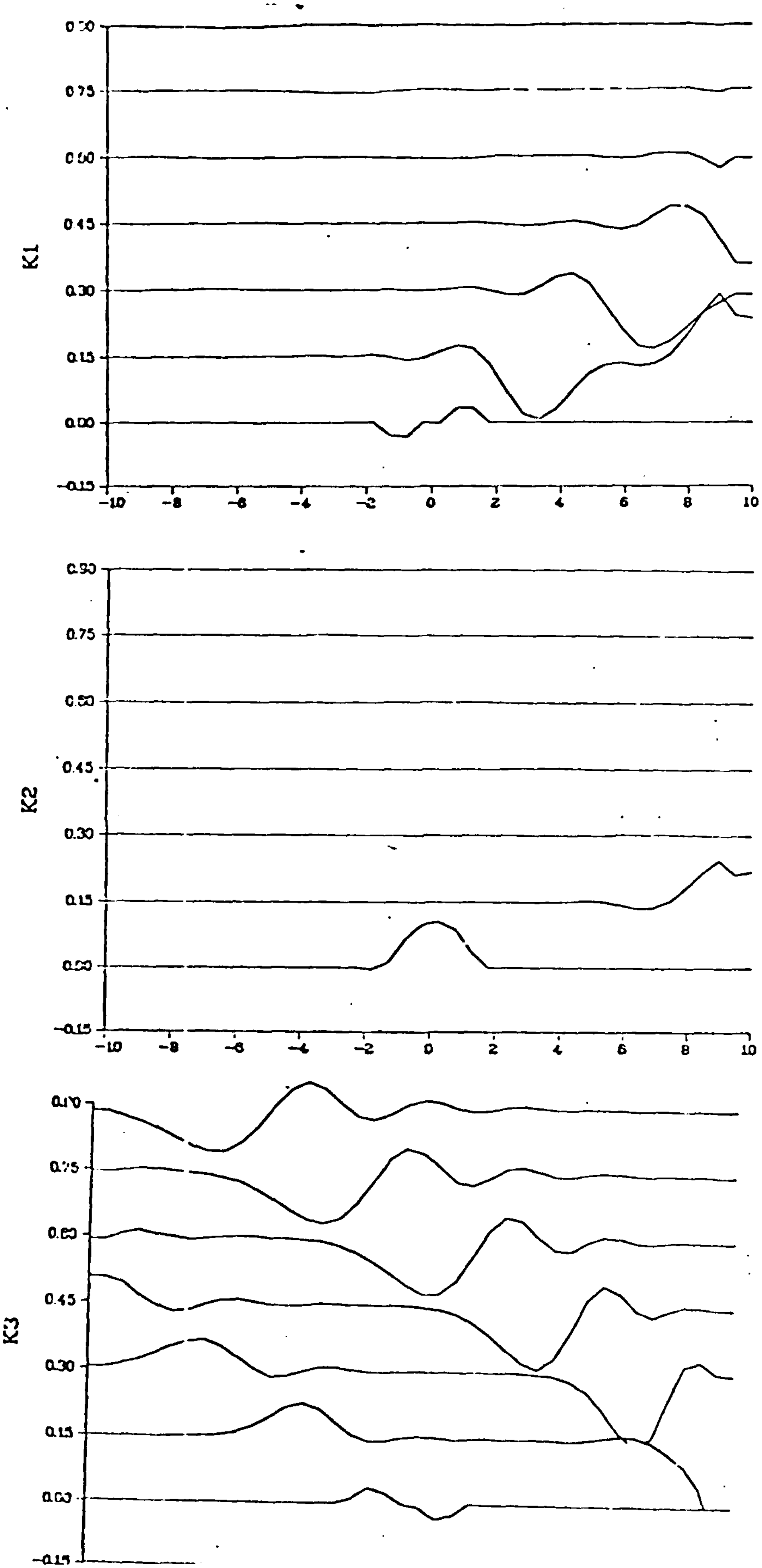


FIGURE 6
A square wave in W_2, reflecting W_3.

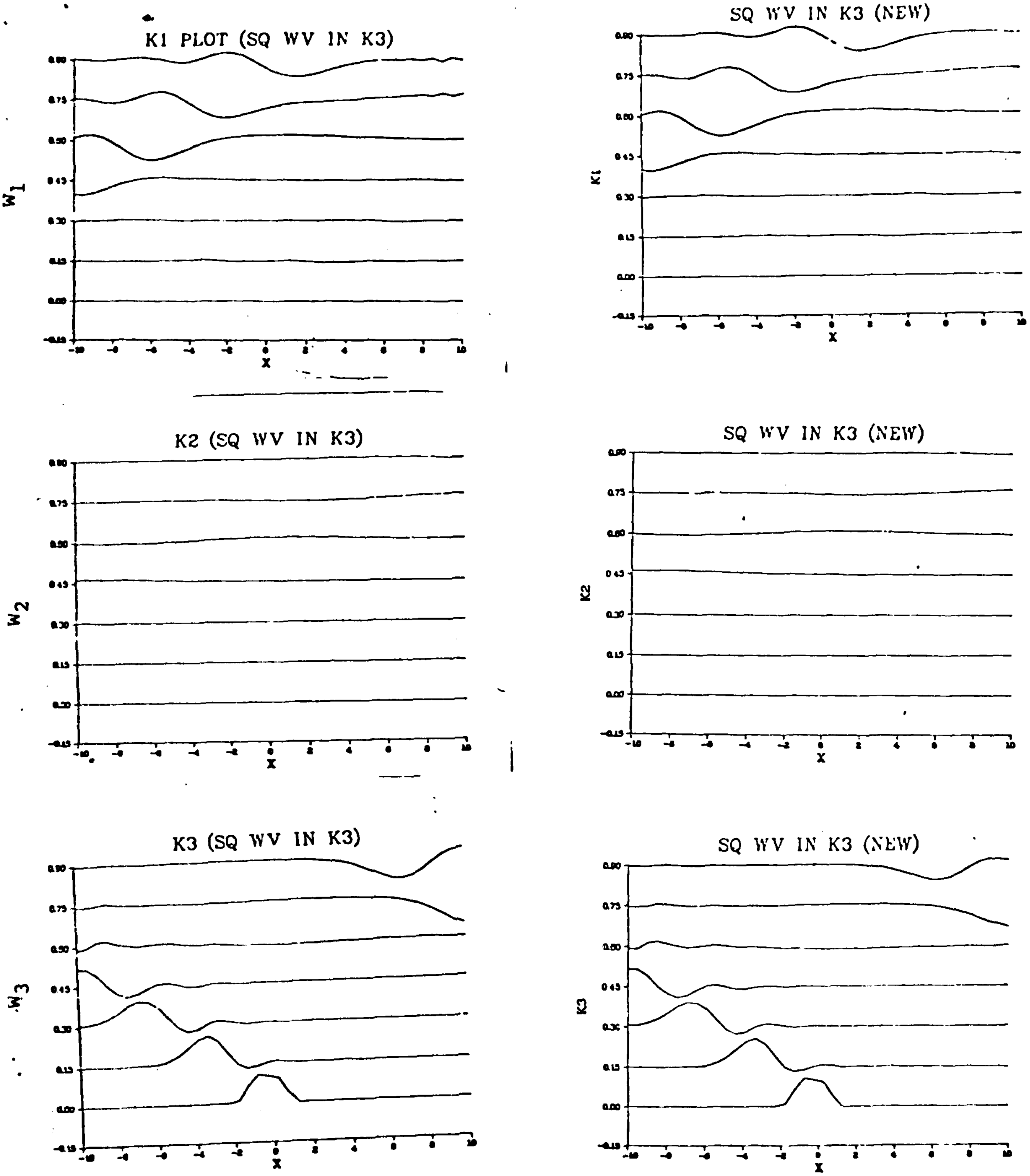


FIGURE 7
Reflecting boundary conditions at both ends. A wave in W_3 travels upstream, causing reflections in W_2 and W_1. W_2 travels downstream, and we see a reflection causing a new disturbance in W_3.

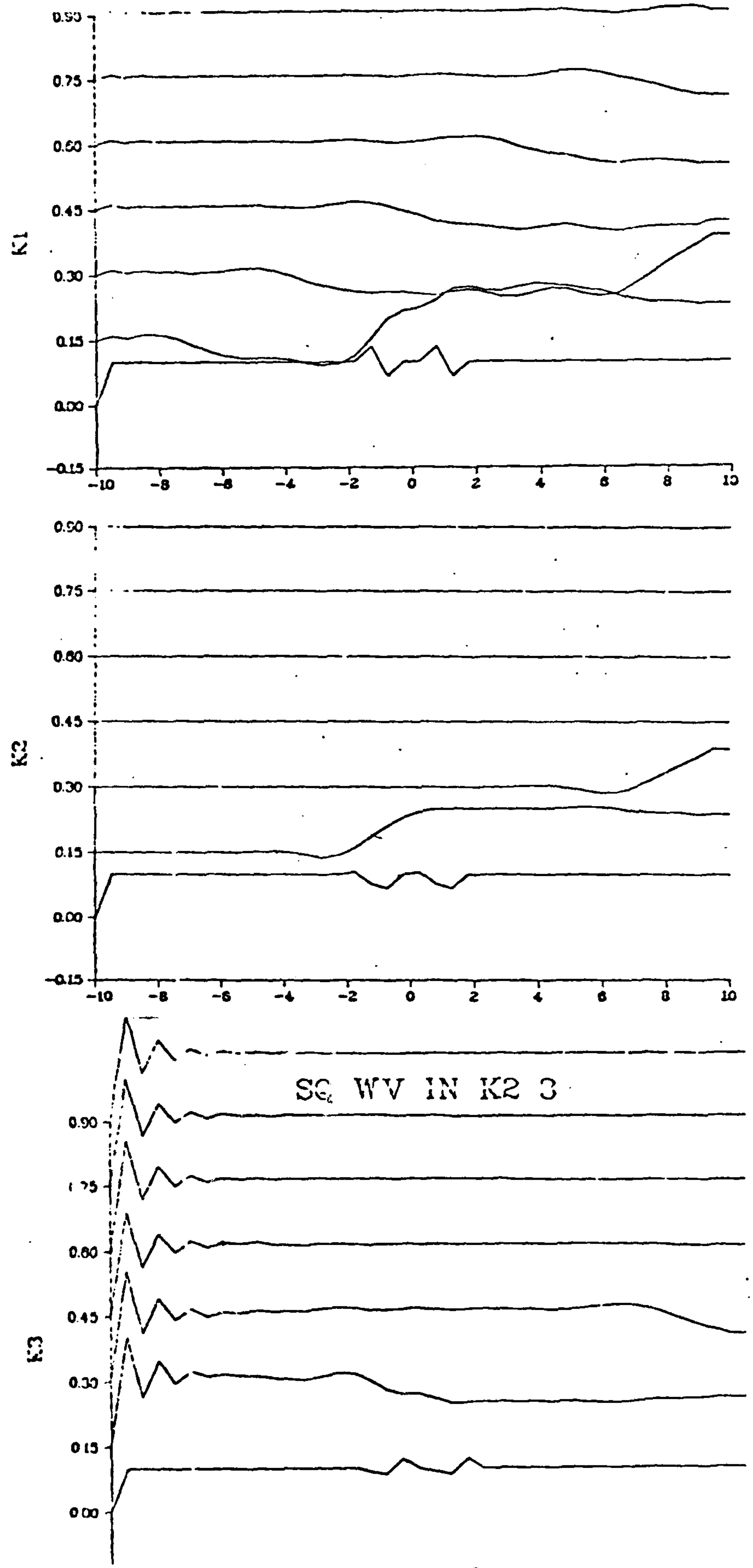


FIGURE 8
Overspecification gives rise to sharp spikes at the boundary, in this case at the inflow.

TWO DIMENSIONAL RESULTS

Having now understood the phenomena which can occur when calculations are made with one space dimension, we now consider the vastly more complicated situation of two space dimensions.

In this situation, we shall solve the Navier Stokes equation (1), again by the standard MacCormack A.D.E. scheme on a 20 x 20 grid, which is a very simplified model of a wind tunnel. We shall continue to consider flow close to the free stream flow previously studied in the one dimensional case. The physical values are given earlier. Figure 8 shows the geometry of the situation. The fluid is flowing in at the top right corner of the grid and flowing out at the bottom left.

We have three distinct types of boundaries to consider. We have the inflow and outflow boundaries (as before) and in addition, two sidewall boundaries, where the fluid is flowing parallel to the boundary.

We shall continue to impose one dimensional boundary conditions of the type given in the first section on the inflow and outflow, along with the additional condition $v = 0$. This says the fluid flow is one dimensional at the inflow and outflow, and seems physically reasonable.

When we come to the artificial sidewall conditions we must undertake another one-dimensional analysis. Thus, we assume all variables are constant in the x-direction and variation only takes place in the y-direction.

This leads to the set of equations

(6) $$V_t + AV_y = 0$$

where

$$\vec{V} = \begin{bmatrix} p \\ u \\ v \\ e \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & u & 0 \\ \frac{u^2}{2}(\gamma-1) & -u(\gamma-1) & 0 & \gamma - 1 \\ 0 & 0 & \frac{e}{e}\gamma - \frac{u^2}{2}(\gamma-1) & 0 \end{bmatrix}$$

Freezing the coefficients of A in (6), we substitue

$$T_1 = P + \frac{p}{co^2}$$

(7) $$T_2 = v + \frac{1}{\rho_o c_o} p$$

$$T_3 = -v + \frac{1}{\rho_o c_o} p$$

$$T_4 = u$$

Equation (6) is then transformed to

$$T_{1t} = 0$$

(8) $$T_{2t} + c_o T_{2y} = 0$$

$$T_{3t} - c_o T_{3y} = 0$$

$$T_{4t} = 0$$

Thus, if we only consider deviations transverse to the free stream flow, we have on the basis of the linearized model, four non-physical variable T_1, T_2, T_3, T_4, two of which u and entrophy T_1 move with zero velocity in the y direction, one of which, T_2, moves with speed c_o in the positive x direction, and one of which moves with speed c_o in the negative x direction. Thus one dimensional theory predicts that at the sidewall, we should impose

$y = 0$	$y = L$
$T_{1y} = \text{const}$	$T_{1y} = 0$
$T_{3y} = 0$	$T_3 = \text{const}$
$T_{4y} = 0$	$T_{4y} = 0$

This, together with the inflow and outlfow conditions gives the following set of boundary conditions

Inflow

$$v_1 = v_\infty$$

$$p_1 = \frac{1}{2} [p_2 + \rho_o c_o (k_3 - u_2)]$$

$$u_1 = \frac{1}{2} [k_3 - \frac{p_2}{\rho_o c_o} + u_2]$$

$$\rho_1 = k_1 + \frac{\rho_o}{2c_o} [\frac{p_2}{\rho_o c_o} + k_3 - u_2]$$

where the free stream values are specified in the characteristic variables $k_1 = \rho_\infty - p_\infty / c_o^2$, $k_3 = \mu_\infty + p/\rho_o c_o$ and where the zero-subscript refers to frozen variables

Outflow

$$v_n = v_{n-1}$$

$$u_n = \frac{1}{2} [u_{n-1} + p_{n-1}/(\rho_o c_o) - k_4$$

$$p_n = \frac{\rho_o c_o}{2} [k_4 + u_{n-1} + p_{n-1}/(\rho_o c_o)]$$

$$\rho_n = \rho_{n-1} - \frac{p_{n-1}}{c_o^2} + \frac{\rho_o}{2c_o} [k_u + u_{n-1} + \frac{p_{n-1}}{\rho_o c_o}]$$

Top wall

$$p_j = \frac{\rho_o c_o}{2} [v_{j-1} + \frac{p_{j-1}}{\rho_o c_o} - k_3]$$
$$v_j = 1/2k_3 + 1/2(v_{j-1} + p_{j-1}/\rho_o c_o)$$

$$\rho_j = \frac{\rho_o}{2c_o} (v_{j-1} + \frac{p_{j-1}}{\rho_o c_o} - k_3) + \rho_{j-1} - \frac{1}{c^2} p_{j-1}$$
$$u_j = u_{j-1}.$$

Bottom wall

$$u_1 = u_2$$
$$v_1 = 1/2[v_2 + k_2 - p_2/(\rho_o c_o)]$$
$$p_1 = \frac{\rho_o c_o}{2} [-v_2 + k_2 + p_2/(\rho_o c_o)]$$
$$\rho_1 = \rho_2 + (\rho_o/2c_o) [-V_2 + k_2 - p_2/\rho_o c_o]$$

where the free stream values are used to specify the characteristic combinations

$$W_2 = v_\infty + \frac{p_\infty}{\rho_o c_o} \qquad k_3 = v_\infty - p_\infty/\rho_o c_o.$$

DISCUSSION OF TWO-DIMENSIONAL COMPUTATIONS

We now discuss a series of computations which illustrate the applications of the boundary conditions we have developed, as well as their short comings. To illustrate varying boundary conditions, we shall consider the experiment where initial conditions are imposed to create a 10% error in T_2, (the left-ward moving variable), whereas T_1, T_3, T_4 are at the correct free stream constants. The ten percent error is imposed for all x at five central y grid values. Thus, if the linear model was exactly accurate, the square wave would exit in the positive y-direction with velocity c_o.

In figure 9, this is shown happening. Figure 9 has Neuman boundary conditions imposed at the inflow and outflow, to illustrate the case of flow in an infinite region. Thus, there is no contribution to the deviation from free-stream from the inflow and outflow. The picture looks almost identical to the one-dimensional plots where the y-variable was ignored. There is some overshoot visible but the wave motion is as predicted and the disturbance in the variable $u + \frac{1}{\rho_o c_o} p$ exits uneventfully, leaving some trailing disturbances which exit in their turn. We now impose boundary conditions at the inflow and outlfow. These boundary conditions correspond to the one-dimensional conditions developed in the earlier sections, plus, in addition, the condition $v = 0$ at the inflow, and $v_x = 0$ at the outflow.

Figure 10 shows the influence of the conditions at the inflow and outflow. The wave does not continue as a one-dimensional wave but becomes smaller near the inflow and outflow as the correct values being imposed at these boundaries makes itself felt. Nonetheless, the picture sharply resembles the pure one-dimensional flow picture of figure 9. There is a small amount of non-linear coupling with the other variables, but it is not significant enough to show up on the plots.

The sidewall boundary conditions used in these plots were the non-reflecting characteristic boundary conditions of the last section.

EFFECT OF REFLECTING BOUNDARY CONDITIONS

We now consider the effects of other sidewall boundary conditions. Figure 11 and 12 show the effects on T_3 if

reflecting boundary conditions are imposed on the right sidewall. We impose the conditions $u = u_\infty$, $e = e_\infty$, $\rho = \rho_\infty = 0$ and $v_y = 0$. In terms of the variable T_i, this imposes the condition $T_2 + T_3$ = const. which ought cause a reflected wave in T_3 as the T_2 wave exits. Figure 10 corresponds to figure 8. One can see that all variations are in the y-direction and just as in figure 8, we had the T_2 wave exit at iterations 50-80 so figure 10 shows a corresponding T_3 wave entering and propogating into the medium.

Figure 11 shows the corresponding picuture for figure 9, when the characteristic boundary conditions are applied at the inflow and outflow. The motion shows the influence of the inflow and outflow boundaries, so that at some stage in the future one would expect convergence to the free stream values, but this process is substantially showed.

EFFECTS OF OVERSPECIFIED BOUNDARY CONDITIONS

Figure 13 shows the interaction of the transverse wave with the boundary when all physical variables are specified at their free-stream values. Note the presence of the jagged spikes as soon as the wave reaches the boundary. This is strikingly similar to the situation of figure (8) in the one-dimensional calculation.

In Figure 14, the situation is almost exactly the same, the principle difference being that the overspecified variables are slightly different on the two sidewalls. This corresponds to the situation where the free-stream values are not known exactly but only approximately. Notice that in the overspecified case of

figure 14, much larger errors are introduced. These errors only seem to be present near the boundary but in more complicated geometries might well propagate back into the region.

As an example of more serious problems with overspecification figure 14 shows what ahappens if an initial disturbance of 10% is made in T_2 only. On the left is shown the wave beginning to exit, uneventfully in the case of our recommended boundary conditions. On the right is shown the large errors occurring when the boundary is overspecified. Note the presence of the large spikes, similar to the one-dimensional setting of figure 7.

We conclude with a stiking example of how one-dimensional the flow can be in the entropy variable T_1.

Figure 16 shows the effect of square wave deflection of 10% imposed uniformly in the x-direction. Note how there is absolutely no movement in the y-direction whatsoever, although as the wave exits, there is predictably overshoot visible at the end of the wave.

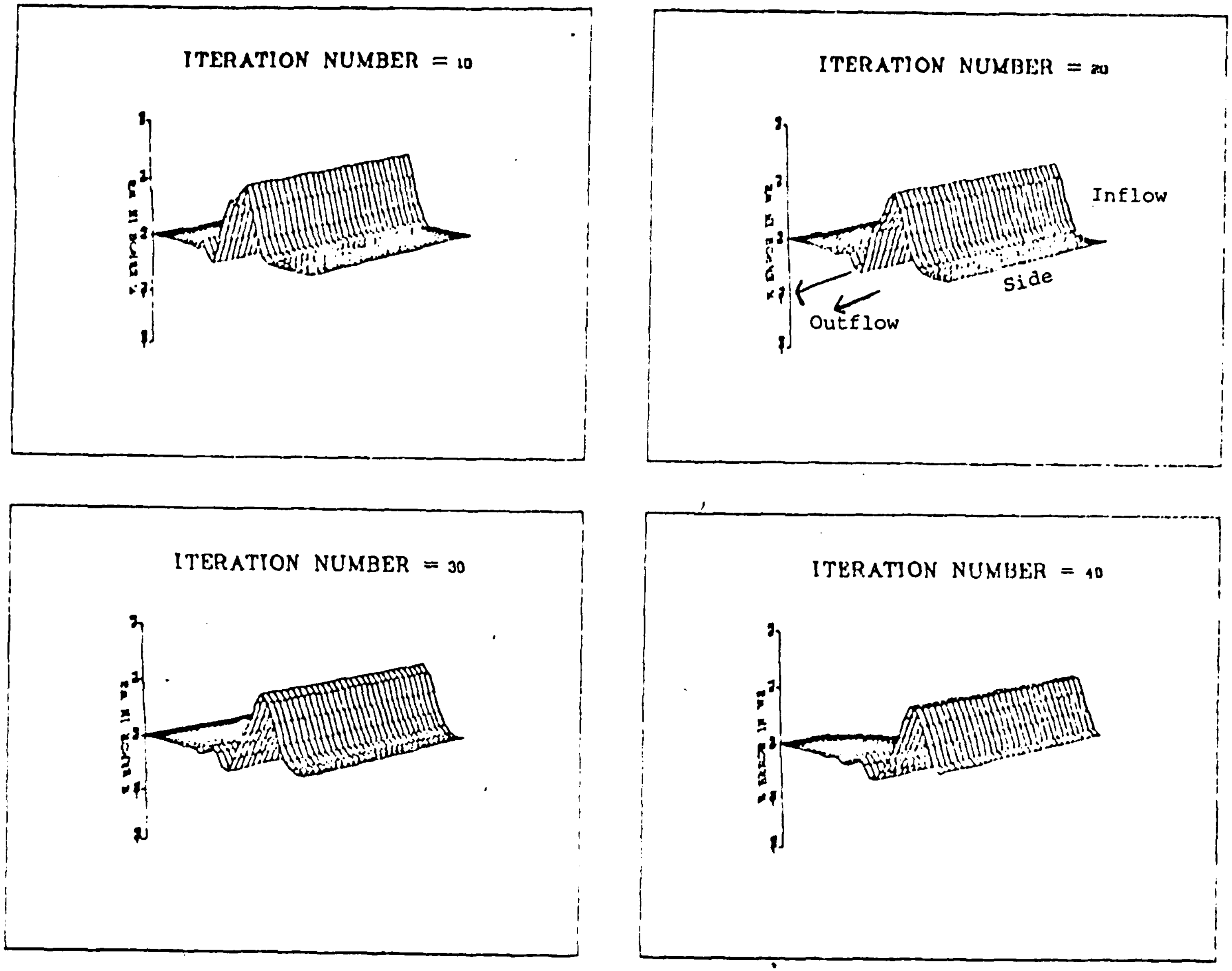


FIGURE 9(a)
Right running characteristic wave begins to exit

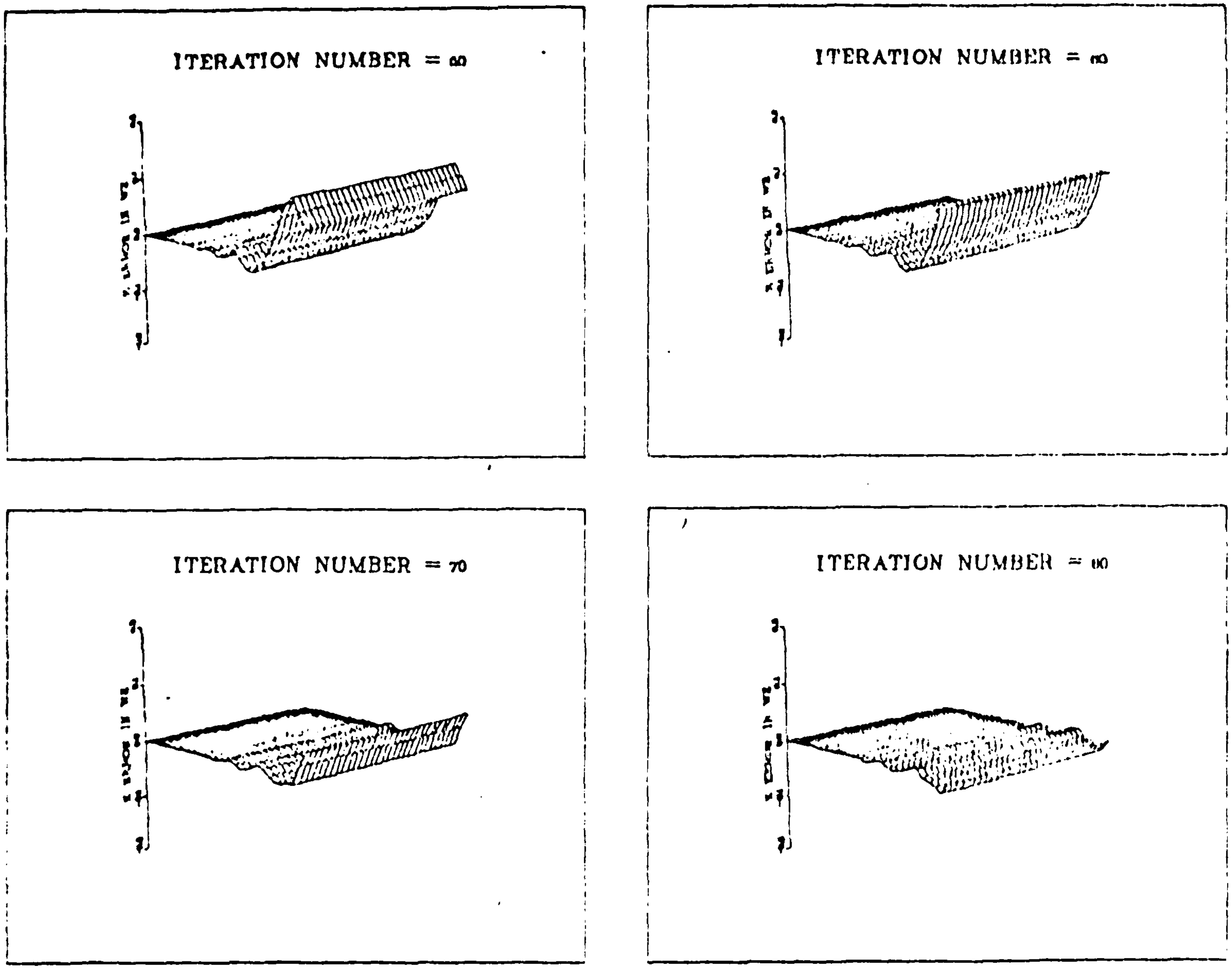

ITERATION NUMBER =
ITERATION NUMBER =
ITERATION NUMBER = 70
ITERATION NUMBER =

FIGURE 9(b)

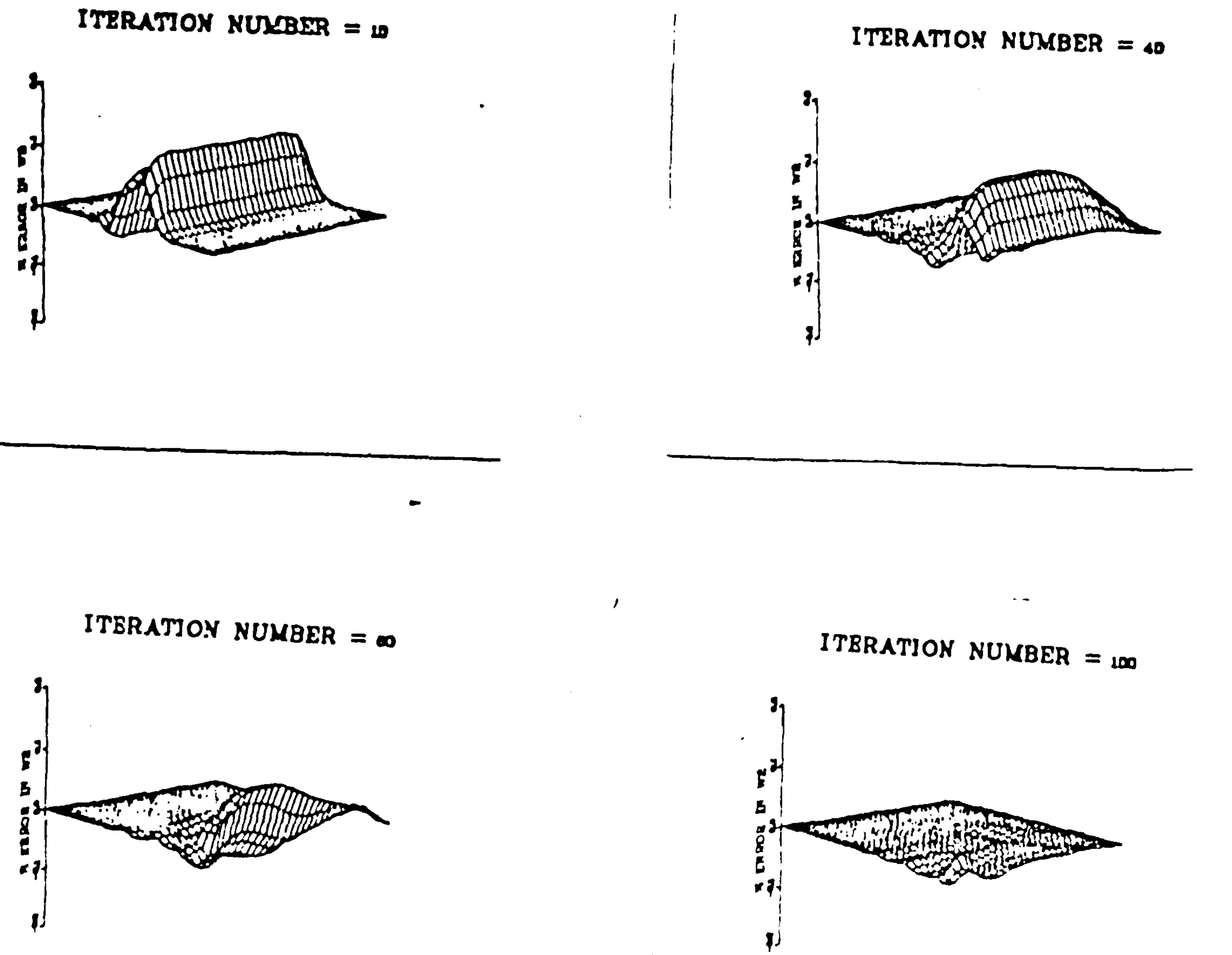

ITERATION NUMBER = 10
% ERROR IN WB
ITERATION NUMBER = 40
% ERROR IN WB
ITERATION NUMBER = 80
% ERROR IN WB
ITERATION NUMBER = 100
% ERROR IN WB

FIGURE 10

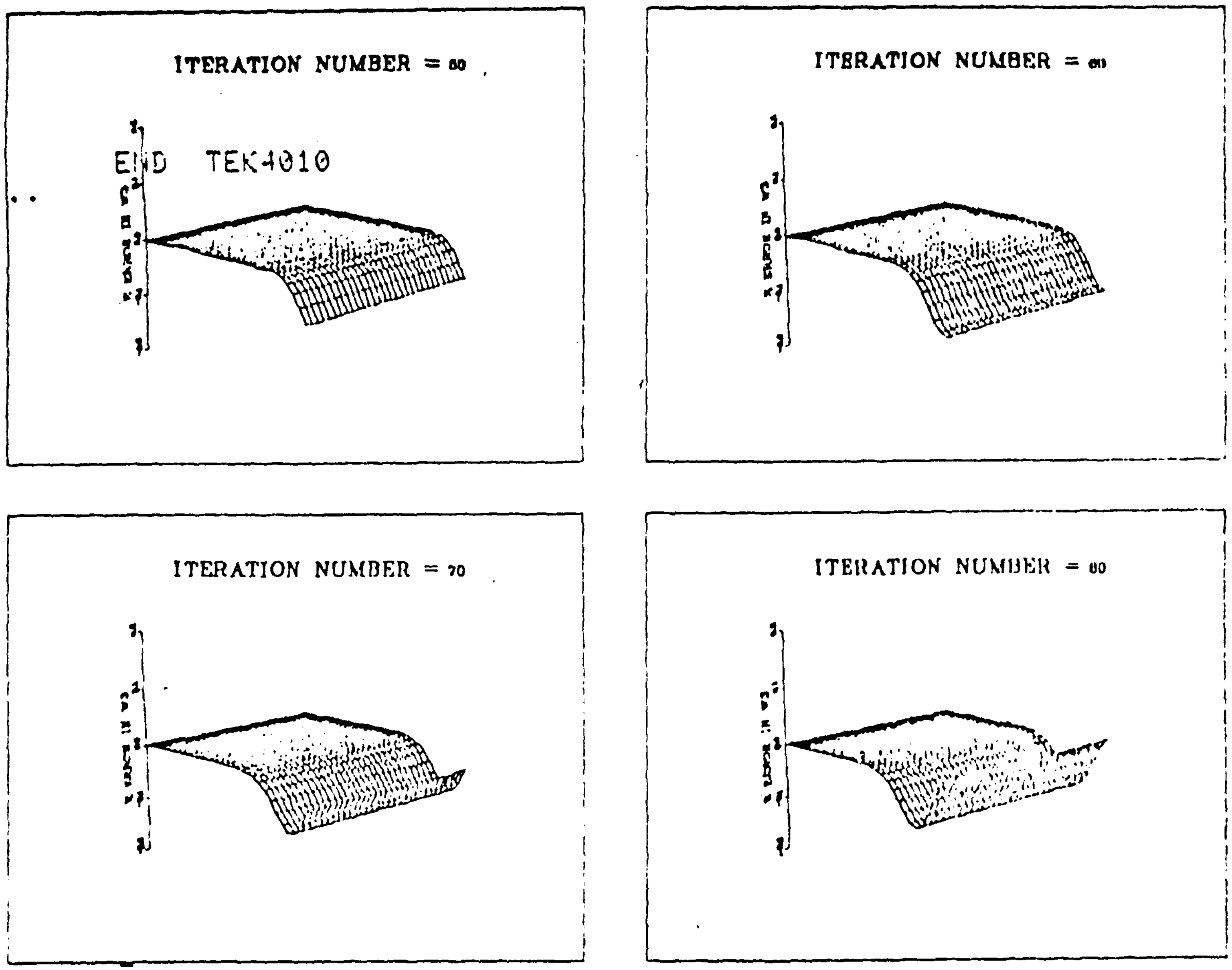
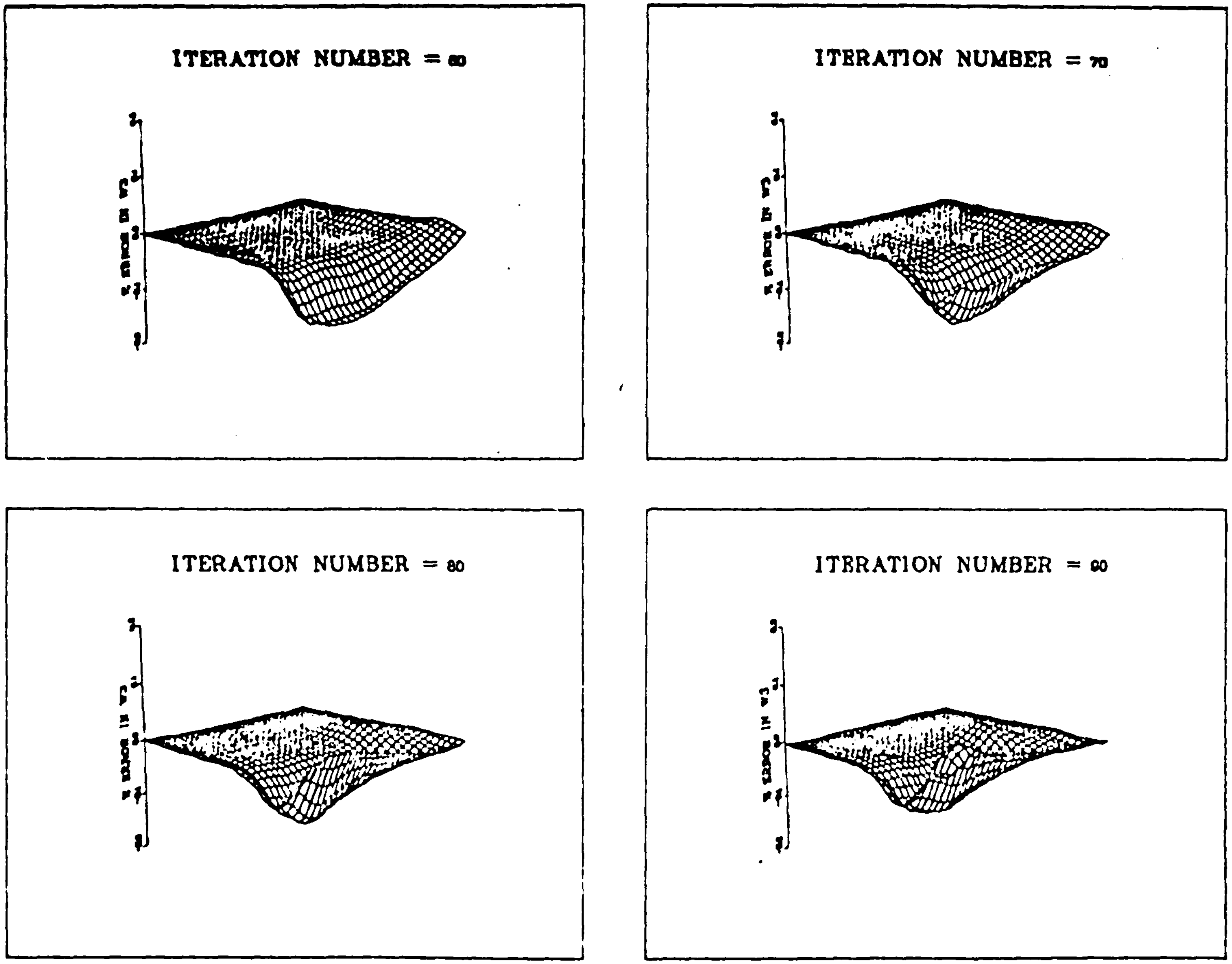


FIGURE 11
With reflecting boundary conditions at sidewall,
a running wave appears as the wave in figure 1(a) exits 1-



ITERATION NUMBER = 60
ITERATION NUMBER = 70
ITERATION NUMBER = 80
ITERATION NUMBER = 90

FIGURE 12

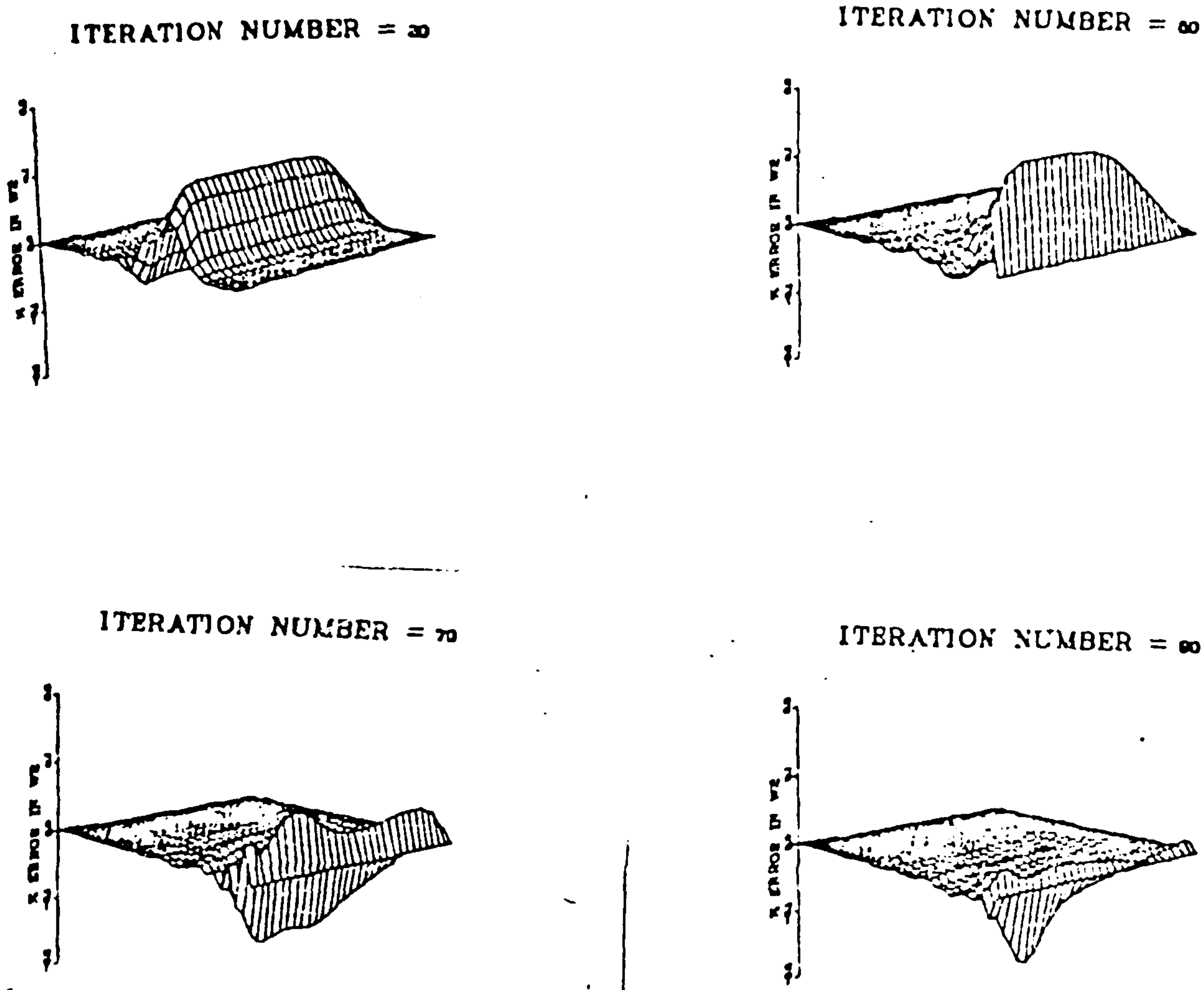


FIGURE 13
A left running wave exits through an overspecified boundary-compare with figure 8 (overspecified 1-D) and figure 10.

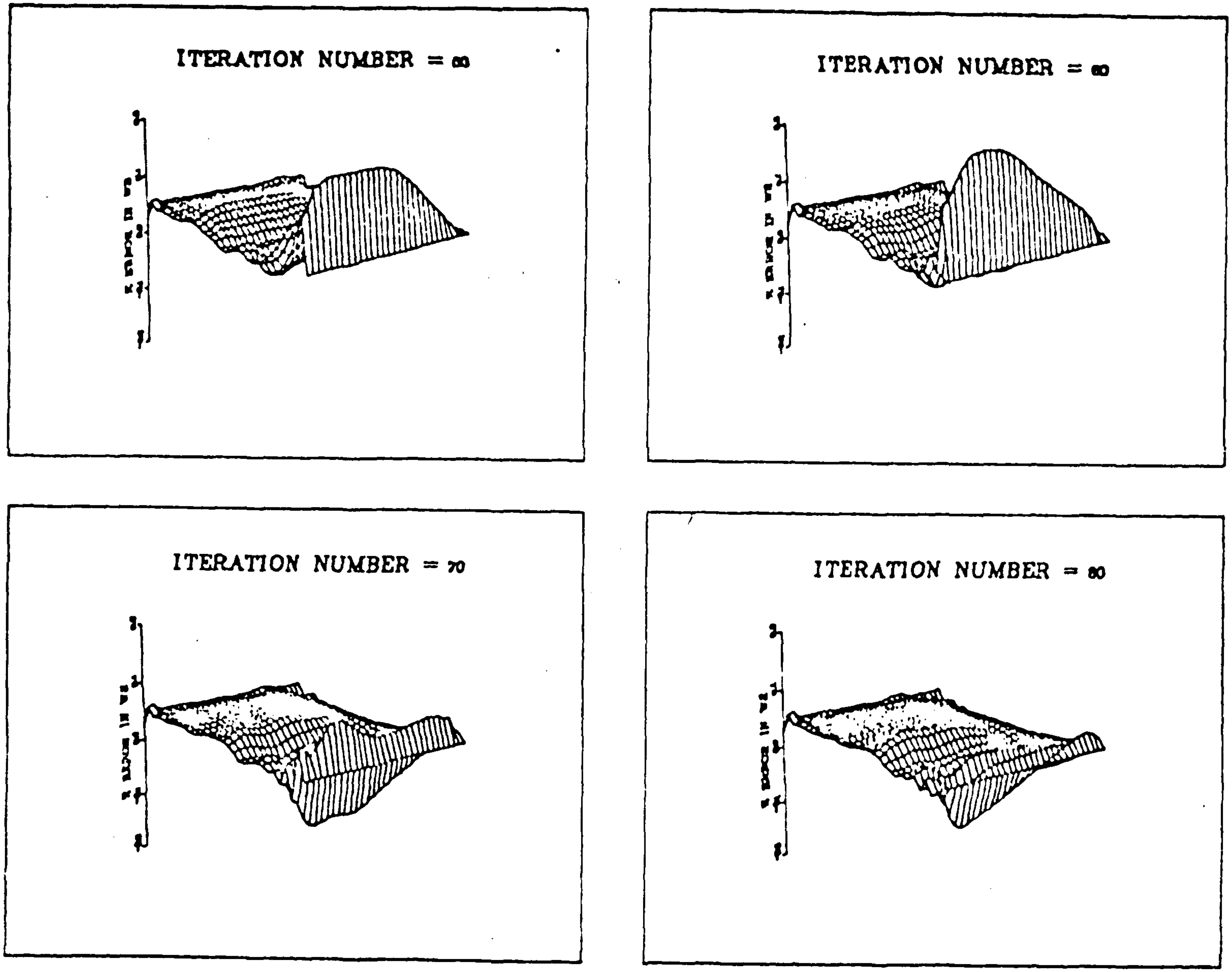


FIGURE 14
Another example of overspecification introducing
large spikes-compare with figure 13.

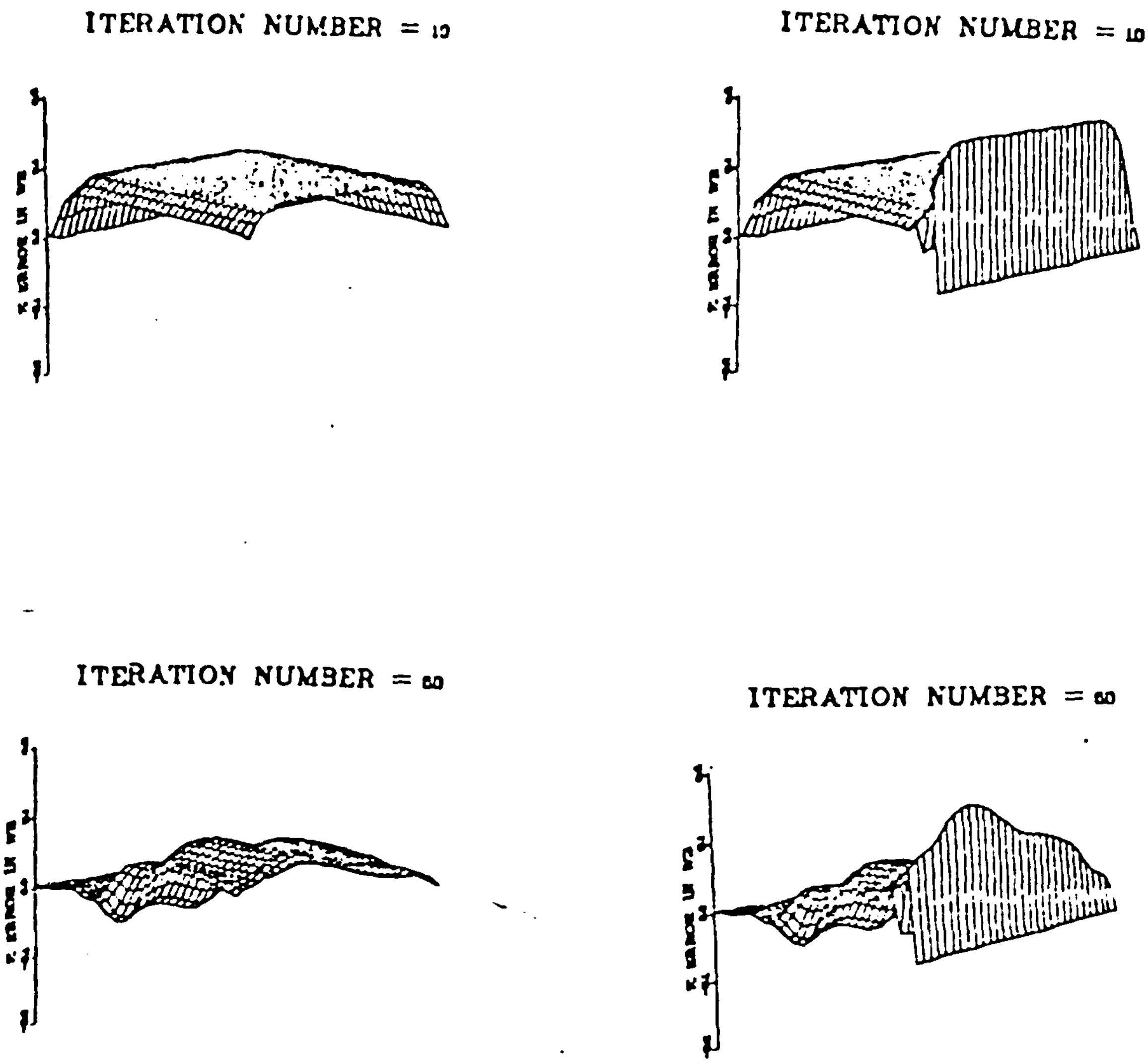


FIGURE 15
Showing a uniformly wrong disturbance in T_2 exiting
a) on the left, with characteristic B.C.'s.
b) on the righrt, with overspecified B.C.s.

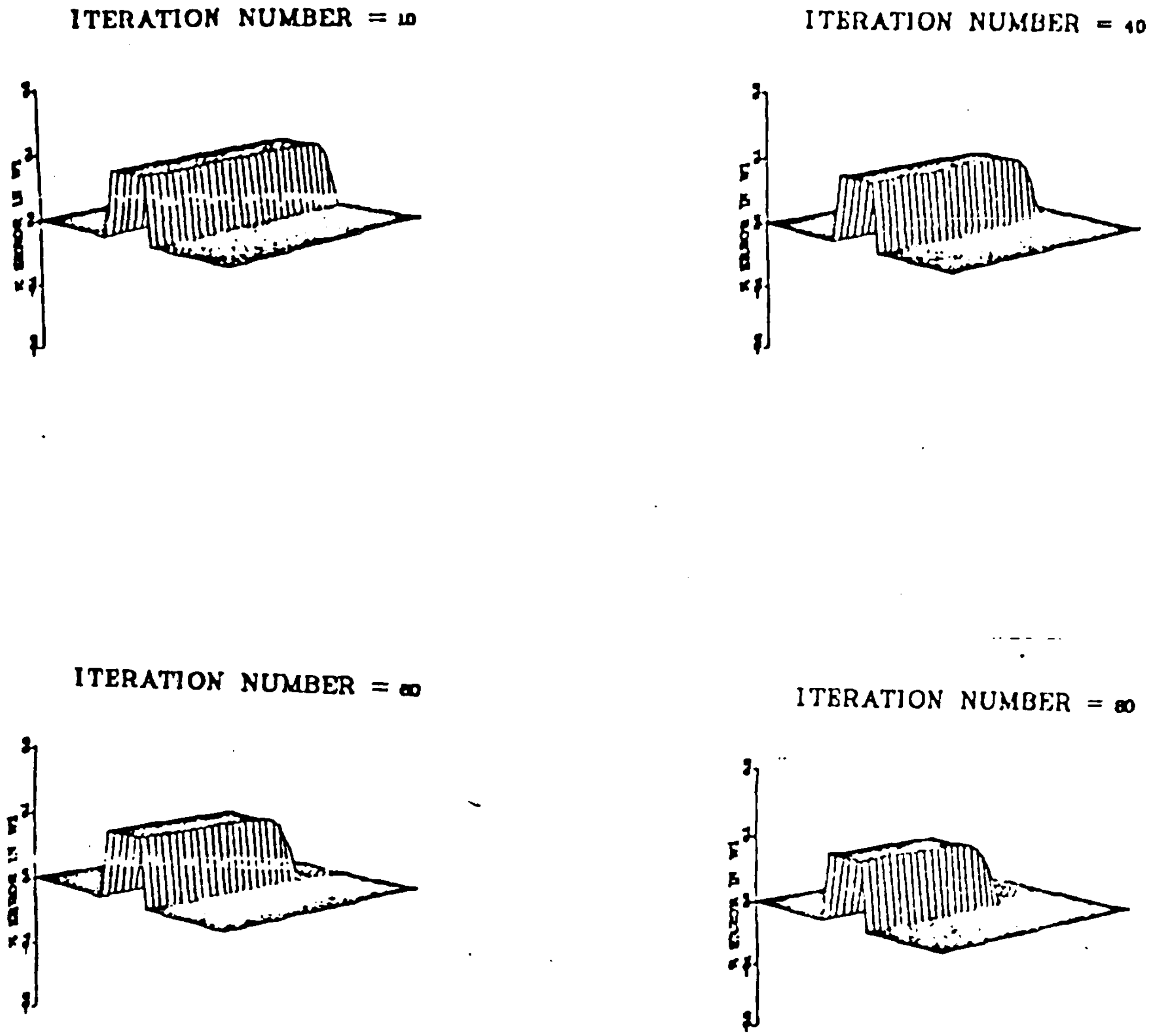


FIGURE 16
Showing the surprising lack of transverse movement in the eutropy variable T_1.

CONCLUSIONS

The results of these numerical experiments show that there are four distinct artificial non-reflecting boundaries which contain free-stream information. These are the inflow, outflow, top-sidewall and bottom-sidewall. Each of these boundaries must be treated differently, or errors are introduced into the system. The worst errors are those associated with over-specification and reflection.

Other more complicated phenomena are present when different types of initial disturbances are present. All confirm the basic lessons of this section, although introducing some complications of their own. At the moment, these other results are not fully understood and call for more experimentation.

RECOMMENDATIONS

This report raises many interesting questions. The most obvious is that we have investigated flow either perpendicular to, or parallel to an artificial boundary, and the boundary conditions required. However, in many applications, to conserve the number of grid points, the boundaries can be at other angles to the expected free-stream flow. It would be desirable to deduce boundary conditions where the flow is at an angle α to the artificial boundary, either entering or exiting the region. It is clear that the conditions, including, as they would have to, all conditions discussed here, would be quite complicated.

Possible improvements to be made in the existing boundary conditions involve the improvement at the corners. There is some

evidence of overspecification in the larger errors that occur there.

It would be interesting to compare these boundary conditions with more complicated problems and possible oscillatory situations.

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